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NOISE MODELING AND RELIABILITY OF BEHAVIOR PREDICTION FOR MULTI-STABLE HYDROELASTIC SYSTEMS

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ABSTRACT

This paper reviews results of experiments conducted on a simple multi-stable hydroelastic (galloping) oscillator. These results show that noise may cause a multi-stable hydroelastic system to exhibit chaotic behavior, and that in some instances such behavior cannot be predicted reliably unless noise effects are carefully accounted for. We then present results of a theoretical investigation of a simple, paradigmatic multi-stable system, the Duffing-Holmes oscillator. The results of this investigation show that for the system being considered, noise promotes the occurrence of chaotic behavior associated with Smale horseshoes. This theoretical investigation is the first phase of an effort to develop analytical tools for predicting reliably the potential for chaotic behavior of actual hydroelastic systems such as deep-water compliant platforms.

INTRODUCTION

Overlooking certain types of dynamic behavior can constitute a gross design error with possibly disastrous consequences. It is such an a design error that led in 1836 to the collapse by flutter of the Brighton Chain Pier bridge. Neither this precedent nor the development of flutter theory by Theodorsen (1935) were understood by the bridge design community, and a similar design error led to the well-known collapse in 1940 of the Tacoma-Narrows bridge.

Thompson et al. (1984) noted that experiments and numerical studies conducted on compliant offshore structures can reveal the possible occurrence of unexpected types of dynamic behavior, including deterministic chaos. In those studies the chaotic behavior of the system of interest occurred in a hydroelastic system excited by periodic loads. More recently, experiments and numerical studies performed on a simple, paradigmatic hydroelastic system showed that irregular behavior involving catastrophic jumps between distinct regions of phase space could also be induced by the noise excitation of a multi-stable system (Simiu and Cook, 1991, 1992). It is this finding that motivated the present work.

Indeed, new types of compliant offshore structures are being envisaged that can be anticipated to exhibit increasingly complex nonlinear behavior, including multi-stability. The hydroelastic studies just alluded to are therefore likely to be relevant in this context. This suggests the need to develop tools for describing and understanding dynamic effects due to the presence of noise. Numerical simulations can be helpful for this purpose. However, as pointed out in the paper, the numerical simulation approach has a number of practical limitations. It it is reasonable to expect that theory could overcome some of these limitations, offer valuable fundamental insights, and help to develop reliable prediction capabilites.

Mathematically, a distinction has been made in the literature between chaotic (i.e., irregular) "basin-hopping" behavior with jumps induced by stochastic excitation or noise (Arecchi et al., 1983), and behavior exhibiting jumps associated with deterministic chaos (i.e., behavior that entails the formation of Smale horseshoes and, therefore, sensitivity to initial conditions, at least one positive Lyapounov exponent, basins of attraction with fractal dimension, and the existence of a strange attractor with fractal dimension (Guckenheimer and Holmes, 1983)). Experimental and numerical results (Simiu and Cook, 1992) have shown, however, that these two types of behavior can be indistinguishable phenomenologically.

In this paper we present a theoretical investigation which shows that, for a certain class of systems and for certain regions of parameter space, the mathematical distinction between noise-induced (stochastic) chaos and deterministic chaos can in fact be artificial. That is, what appears to be basin hopping caused by any given realization of a noisy process can be in fact a form of chaotic behavior associated with the formation of Smale horseshoes. Basic theory yields in this case necessary conditions for the occurrence of such chaotic behavior, as well as a useful measure of its strength as reflected, for example, in the frequency of the jumps.

We emphasize, however, that at this point our theoretical results are not directly applicable to hydroelastic structures. In spite of the useful insights they provide, they should be viewed as representing a first step toward developments applicable to wider classes of systems, including hydroelastic systems.

Following a brief review of relevant experimental and numerical results, we focus our attention on a class of systems for which available theory allows the development of useful analytical tools. We then present results yielded by those tools, which include the generalized Melnikov function and the phase space flux. We then comment on the usefulness and limitations of the approaches discussed in the paper.

NOISE-INDUCED JUMPS IN A HYDROELASTIC SYSTEM

In this section we review briefly experimental and numerical results obtained in the study of a galloping system. The system consisted of two elastically restrained and elastically coupled horizontal square bars immersed in a uniform horizontal water flow. The upstream faces of the two bars were contained in a vertical plane normal to the flow velocity. Drag wires constrained the bars to oscillate in an arc with large radius tangent to the vertical plane (Fig. 1) — see Simiu and Cook, (1991, 1992) for details.

Any deviation of a bar from its position of equilibrium causes self-excited lift (galloping) forces that result

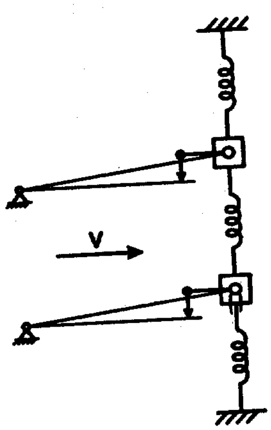

Fig. 1. Galloping device (after Simiu and Cook, 1991).

in an increase of that deviation, that is, the position of equilibrium is an unstable fixed point, so that for small deviations the hydrodynamic damping inherent in the lift is negative. For larger deviations the hydrodynamic damping becomes positive. This limits the amplitude of the oscillation, which thus describes a limit cycle.

In addition to the galloping forces the bars are subjected to forces induced by vortices shed in their wakes. Observations showed that, for relatively low reduced flow velocities, the two bars oscillate in phase, that is, both bars move together up or down. As the reduced velocity grows, in-phase oscillations alternate irregularly with opposite-phase oscillations, where one of the bars moves up while the other moves down. An example is shown in Fig. 2, which depicts displacements (in meters) of the top prism, and Fig. 3, which depicts displacements of both the top and bottom prism for a 5 s interval of Fig. 2.

Motions such as those of Figs. 2 and 3 are reminiscent of the irregular alternation between different oscillatory forms of a forced magnetoelastic beam (Moon

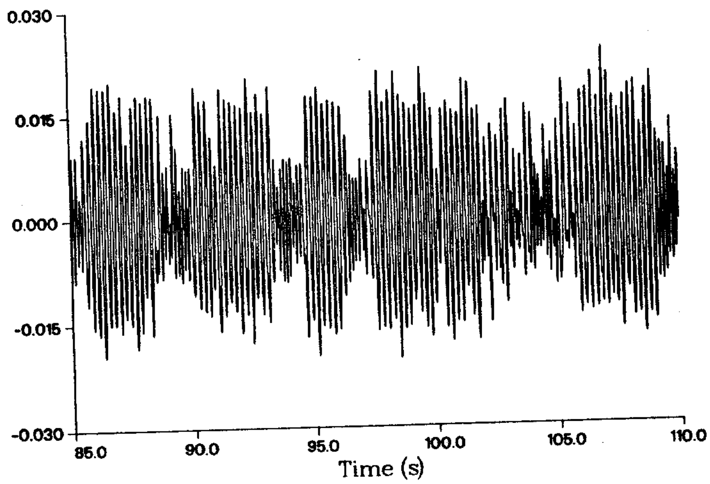

Fig. 2. Observed displacement time history, upper bar.

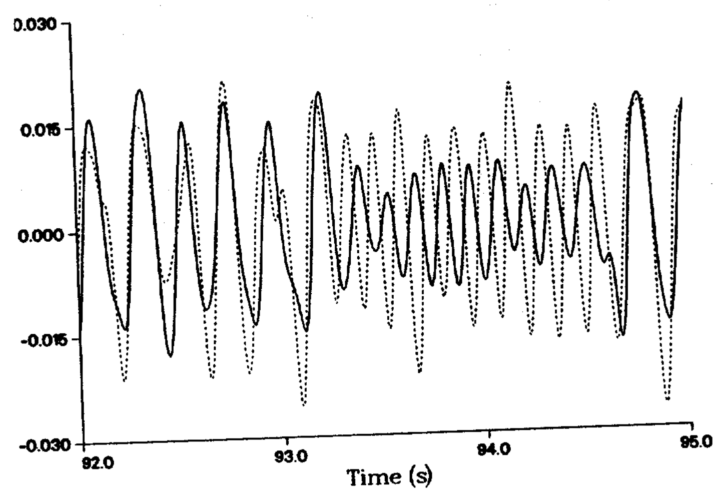

Fig. 3. Observed displacements, upper and lower bar.

and Holmes, 1979) or a forced buckled column (Cook and Simiu, 1991). The motions are irregular, and for any given system the frequency of the alternations increases with the flow velocity. A visual examination of the time histories is not sufficient to reveal whether the irregularity of the motion is due to the randomness of the excitations or to chaoticity associated with Smale horseshoes. For convenience we refer here to the alternations between in-phase and out-of-phase oscillations depicted in Figs. 2 and 3 as irregular jumps.

A dependable numerical simulation of the galloping motions of this system would in principle require the solution of the Navier-Stokes equations. Because such a solution would entail considerable if not insuperable difficulties, engineers resort instead to semi-empirical models for the hydrodynamic loads. Thus, following Parkinson and Smith (1964), the self-excited lift forces are described by nonlinear functions of the angle of attack, with Reynolds-number dependent constant coefficients obtained from measurements under static conditions. This model is clearly imperfect, though for certain applications it yields acceptable results. The vortex-induced lift forces are also described empirically, a commonly used description being given by the so-called lift oscillator model — see, e.g., Simiu and Scanlan (1986, p. 202). In addition, random forces are acting on the bars. These are associated with flow separation (both spanwise and at the bars' ends), oncoming and wake flow turbulence, and irregular flow around ancillary parts. The modeling issue for these random forces is difficult and, in many cases, it has not been resolved satisfactorily to date.

Attempts to reproduce observed motions with irregular jumps by omitting random forces from the hydrodynamic loading model were reported in some detail by Simiu and Cook (1992). (With the random forces absent, the oscillator is modeled by an eighth-order autonomous differential system.) In some isolated cases those attempts were successful, that is, chaotic behavior was obtained by solving the differential system numerically. However, most simulations failed to predict reliably the occurrence of jumps. Simulations were then attempted in which random forces acting on the bars were included. For simplicity this was done by assuming those forces to be modeled by white noise, with intensity chosen by trial and error to yield time histories comparable to those observed in the laboratory. Thus, whereas a hydrodynamic model from which random fluctuations were omitted had neither predictive nor descriptive capabilities, a model in which a crude representation of such fluctuations was included was capable of describing — though not predicting — the observed motions fairly adequately (Simiu and Cook, 1991).

The results thus show, at least for certain types of multi-stable systems, that to predict motions with irregular jumps it is necessary to model the actual hydrodynamic forces, including their random parts. Once this is done, simulations can help to assess the tendency of the system to experience catastrophic jumps induced by random excitation (or noise).

EFFECT OF NOISE ON A MULTI-STABLE SYSTEM: THEORY

Motivated by the results reviewed in the preceding section, we sought to develop a theoretical approach to the prediction of irregular jumps in a nonlinear system. No general theoretical approach to this problem exists. However, for dynamical systems with a global geometry of a type described below, Melnikov theory and the related notion of phase space flux do provide useful theoretical insights. Restricting ourselves to dynamical systems of that type, we study the effects of noise on the system's sensitivity to chaos. This is the first phase in a planned effort to develop analytical tools applicable to structures of interest in ocean engineering.

We now describe our model and briefly introduce some of the fundamentals of Melnikov theory and phase space flux.

Dynamical Model

The systems we consider consist of second-order perturbed differential equations whose unperturbed flows include homoclinic or heteroclinic orbits. For definiteness we focus our attention on a typical example of such a system, the Duffing-Holmes oscillator. Its equation is

$$\ddot{x} = x - x^3 + \epsilon[\gamma\cos\omega t + \sigma G_t - k\dot{x}] \qquad (1)$$

where ϵ is a small number, γ, σ, k and ω are constants, and

$$G_t = [2/N]^{1/2} \sum_{n=1}^{N} \cos(\omega_n t + \phi_n) \qquad (2)$$

where $\{\omega_n, \phi_n; n=1,2,\ldots,N\}$ are independent random variables, $\{\omega_n; n=1,2,\ldots,N\}$ are positive and identically distributed with density Ψ, and $\{\phi_n; n=1,2,\ldots,N\}$ are identically uniformly distributed over the interval $[0,2\pi]$. N is a fixed parameter of the model assumed to be finite, though it may be arbitrarily large.

Shinozuka Noise.

The process G defines Shinozuka noise (Shinozuka, 1971). It is uniformly bounded with zero mean and unit variance. For large N, G is nearly Gaussian and has one-sided spectral density $2\pi\Psi$. Thus by properly choosing the density of ω_n, Shinozuka noise can have any specified spectral density. We choose Shinozuka noise over other types of noise commonly employed in engineering applications, for example Nyquist noise (Rice, 1954), because it is uniformly bounded. This property is essential for the application of Melnikov theory.

Melnikov function.

According to Melnikov theory, a necessary condition for the occurrence of chaos in a system such as Eq. 1 is that its generalized Melnikov function have simple zeros (Wiggins, 1988, p. 463), (Arrowsmith and Place, 1990, p. 174). For Eq. 1 the generalized Melnikov function has the expression (Wiggins, 1988, p. 463):

$$M(t_1,t_2) = -3k/4 + S(\omega)\sin(\omega t_1+\phi_o) + Z_{t2} \qquad (3)$$

with

$$S(\omega) = (2)^{1/2}\pi\omega\,\mathrm{sech}(\pi\omega/2) \qquad (4)$$

and

$$Z_{t2} = \sigma N^{-1/2} \sum_{n=1}^{N} S(\omega_n)\sin(\omega_n t_2+\phi_n) \qquad (5)$$

From the general form of the expression for the Melnikov function (Arrowsmith and Place, 1990) it follows immediately that Z_{t2} is the result of the convolution of $G(t_2)$ with $x_s(-t)$, where

$$\{x_s(t),\ \dot{x}_s(t)\} = \{[2]^{1/2}\mathrm{sech}t,\ -[2]^{1/2}\mathrm{sech}t\ \tanh t\}$$

are the phase plane coordinates of the homoclinic orbit of the unperturbed Duffing-Holmes oscillator (Wiggins, 1990, p.513). Consequently $x_s(-t)$ may be interpreted as an impulse response function. (A similar observation can be made for the second term in the r.h.s. of Eq. 3.) Since the expectation of $G(t_2)$ is zero, so is the expectation of Z_{t2}. The variance of $M(t_1,t_2)$ is easily obtained by noting that the spectral density of Z_{t2} is equal to $S^2(\omega)$ times the spectral density $\Psi(\omega)$, since $S^2(\omega)$ may be interpreted as the square of the modulus of the transfer function corresponding to the impulse response function $x_s(-t)$. Thus,

$$\sigma_Z^2 = \sigma^2 \int_{-\infty}^{\infty} S^2(\omega)\Psi(\omega)d\omega \qquad (6)$$

From Eq. 5 it follows that the random variable Z_{t2} is the sum of bounded, independent, identically distributed terms. For large N, its distribution is therefore nearly Gaussian with zero mean and variance σ_Z^2.

Flux Factor.

The average space flux is a measure of the phase space transport that makes possible the occurrence of such a jump (Wiggins, 1990). It reflects the strength of chaotic behavior and, therefore, the frequency of the jumps. The flux may be viewed as a measure of the probability that an orbit will cross the pseudo-separatrix that separates regions of phase space associated with the potential wells of the unperturbed system. As shown by Beigie, Leonard and Wiggins (1991), for small ϵ the average space flux is, to first order, $\epsilon\Phi$, where the flux factor Φ is related to the Melnikov function as follows:

$$\Phi = \lim_{T \to \infty} (1/2T) \int_{-T}^{T} M^+(s, \theta_1, \theta_2)ds \tag{7}$$

In Eq. 7 $\theta_i = t_i + s$ $(i = 1,2)$ and M^+ denotes the positive part of the function M. It can be shown (Frey and Simiu, 1992) that, for the system defined by Eqs. 1 and 5, the limit in Eq. 7 exists, and that for large N,

$$\Phi = E[(\sigma\sigma_Z\Lambda \;\; + \gamma B - 3k/4)^+] \tag{8}$$

where E denotes expectation, Λ is the standard Gaussian variable, $B/S(\omega) = b$ is a random variable, independent of Λ, with density

$$f(b) = \frac{1}{\pi} \frac{1}{(1-b^2)^{1/2}}, \qquad -1 < b < 1 \tag{9}$$

and g^+ denotes the positive part of the function g. For any given spectral density of the Shinozuka noise, σ_Z can be obtained from Eq. 6. Then, for any given set of values σ, γ and k, Φ can be obtained from Eq. 8 by numerical integration.

Equations 3 and 8 show that the excitation by noise and the periodic excitation contribute in similar ways to the fact that the system is chaotic. Thus, the system may experience chaos characterized by a Melnikov function with simple zeros and given flux Φ owing to an excitation by (1) a periodic or quasiperiodic function (in the case we examined, a harmonic function), (2) a realization of a stochastic process, or (3) a combination of (1) and (2). Any fundamental distinction between jumps induced by noise and deterministic chaos is then erased; the only difference between a harmonic and a non-periodic excitation is that for the former the Smale horseshoes and the strange attractor are fixed in time, whereas for the latter chaos is associated with traveling horseshoes and the strange attractor is time-dependent (Beigie, Leonard and Wiggins, 1991).

Note that the spectral density of the random process affects the flux through its presence in the expression for the variance of the filtered noise process, σ_Z^2 (see Eq. 6). Thus, the effect of the noise excitation depends primarily on the strength of its frequency content in regions of the spectrum where the values of the transfer function $S(\omega)$ are large.

Example. To illustrate the relation between flux and frequency of jumps we consider the Duffing equation with $k = 0.24$, $\omega = 1.4$ and $\sigma = 0$. A bifurcation diagram for this system is shown in Fig. 4. We have in this case

$$\Phi = E[(\gamma B - 3k/4)^+]. \tag{10}$$

The flux increases with the excitation amplitude γ, as

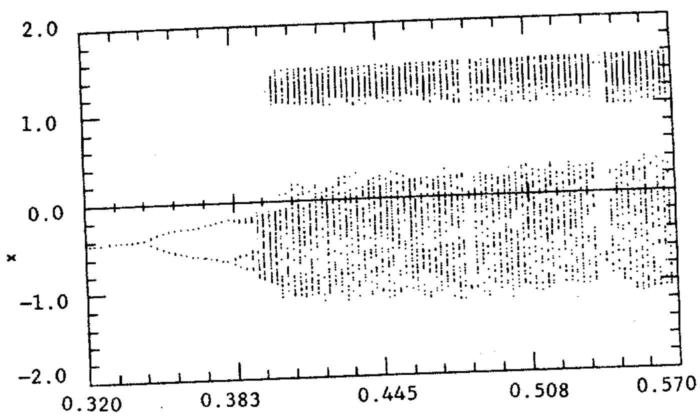

Fig. 4. Bifurcation Diagram (x versus γ).

expected. Figure 5 shows time histories of the displacement x for four values of γ to which there correspond chaotic motions with snapthrough (jumps). (Figures 4 and 5 were obtained by a computer program described by Parker and Chua (1989)).

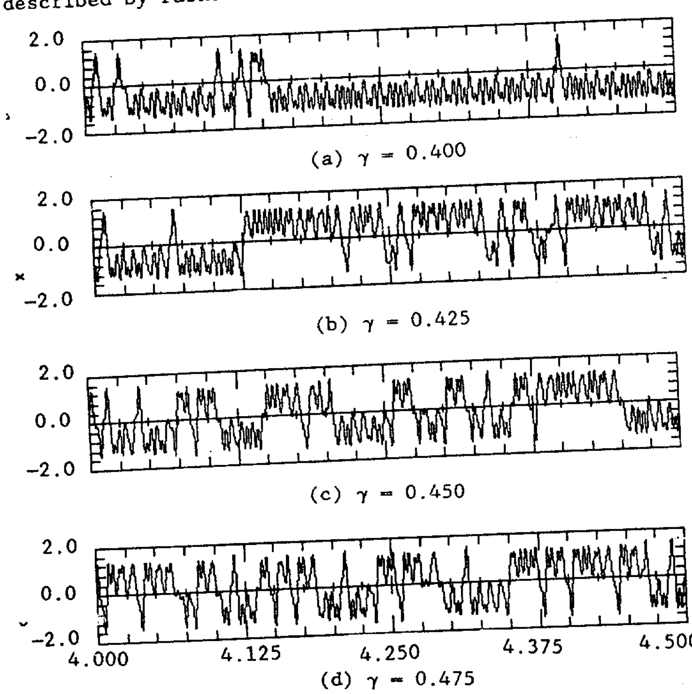

Figure 5. Time histories of displacement x.

Table 1 shows, for each value of γ, the estimated mean escape time (time between successive jumps) in nondimensional units (based on a time interval T = 20,000), and the ratio $\Phi(\gamma)/\Phi(\gamma=0.40)$, which was estimated numerically.

Table 1. Mean Escape Times and Ratios $\Phi(\gamma)/\Phi(\gamma=0.400)$

γ	Mean Escape Time	Ratio $\Phi(\gamma)/\Phi(\gamma=0.400)$
0.400	120	1.000
0.425	55	1.062
0.450	35	1.126
0.475	28	1.187

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DISCUSSION

Theoretical Investigation

The theoretical investigation of the previous section answers the question raised by Bulsara et al. (1991) whether noise raises or lowers the threshold for the occurrence of chaos in a system whose unperturbed counterpart has a homoclinic orbit. Whereas these authors concluded that noise raises that threshold, that is, that the addition of noise makes it harder for chaos to occur, we have shown that the opposite is the case. (For a comment on the flaw in Bulsara et al. (1991) that led to that erroneous conclusion, see Simiu et al. (1991). We note also that the treatment of and the conclusions for the homoclinic and heteroclinic cases are similar). In addition, we have shown that deterministic chaos with given strength can occur under excitation by (1) a harmonic or quasiperiodic function, (2) a realization of a random process, or (3) a combination of (1) and (2). In the latter two cases the irregularity of the response and, in particular, the irregular occurrence of jumps, can be due to the development under the total excitation of a chaotic condition associated with traveling Smale horseshoes, rather than to just the irregularity of the excitation. Finally, we have shown that the spectral density of the noise can have an important role in determining the behavior of the system.

For the class of systems considered, the necessary condition for the occurrence of chaos is related to the behavior of the Melnikov function, and the magnitude of the phase space transport across a pseudo-separatrix (and therefore the systems's susceptibility to the occurrence of jumps) is related to the flux factor function. Those functions are helpful indicators even in the absence of information on the configuration of the basins of attraction and the attractors in phase space. Recent fundamental studies deal explicitly with the relation between basins of attraction and susceptibility to noise effects, see, e.g., Thompson (1989), Soliman and Thompson (1989), Lansbury and Thompson (1990), and Thompson and Soliman (1991).

It should be emphasized that the indicators in question are based on the assumption that, in Eq. 1, ϵ is small. For this reason the Melnikov function criterion provides only a necessary condition, and the flux calculations can only be viewed as very rough approximations. To obtain closer estimates of the flux lengthy numerical calculations would be needed, as described in some detail by Beigie, Leonard and Wiggins (1991).

Even with these limitations, the applicability of the theoretical approach presented is restricted to the narrow class of dynamical systems defined in the preceding section. In particular, the approach is not applicable to complex structures such as compliant offshore platforms. Intuitively, one might surmise that the findings reported here on the role of noise could provide useful qualitative insights applicable to systems other than those dealt with in the preceding section. The extent to which this is the case remains to be established, however.

Numerical Approach

At this time numerical simulations are the only available method for investigating the behavior of actual, complex nonlinear structures under excitation by noise. As noted earlier, the susceptibility of a system to the occurrence of undesirable jumps depends on the configuration of the basins of attraction and the attractors in phase space. The proximity of an attractor to a separatrix increases the susceptibility of a system to noise effects — see, e.g., Soliman and Thomson (1989).

Depicting that configuration can be a prohibitive task even in a deliberately simple system such as the galloping oscillator studied by Simiu and Cook (1991, 1992), for which the dimension of the phase space is eight. For an actual offshore structure the difficulties would be even greater. For this reason numerical simulations may be viewed as a practical, though not fool-proof means of exploring the susceptibility of the system to various types of noise.

A sufficiently large set of initial conditions should be used to minimize the possibility that an attractor of potential significance is missed. An additional practical difficulty can be the lack of information on the spectral density of the random excitation acting on a structure whose fluid dynamic characteristics are not known in detail. Given the potential significance of the noise spectrum, as revealed by theory in the case dealt with earlier, the analyst would be well advised to perform numerical experiments using white noise with various spectral densities.

CONCLUSIONS

Results of experiments and numerical simulations reviewed in this paper suggest that motions with noise-induced jumps can be indistinguishable visually from deterministic chaotic motions. Those results motivated the theoretical investigation of a simple, bi-stable dynamical system with noise excitation. The investigation showed that the two types of motion can in fact belong to the same class not only phenomenologically, but mathematically as well, that is, the motions with noise-induced jumps were shown to be chaotic motions associated with the formation of Smale horse-shoes. A consequence of this finding is that, for one-degree of freedom systems whose unperturbed counterparts have homoclinic or heteroclinic orbits, noise decreases the threshold at which chaotic behavior associated with Smale horseshoes occurs. Finally, it has been shown that the spectral density of the noise excitation can play an important role in determining the system behavior.

The applicability of the results of our theoretical investigation is restricted. Whether the insights they afford on the role of noise can be extended to other types of systems remains to be established.

There are difficulties associated with the numerical exploration of the susceptibility to noise-induced jumps of complex nonlinear structural systems such as deep-water compliant platforms. These difficulties include the need to consider a wide range of sets of initial conditions so that no significant attractors be missed, and the possible lack of information on the spectral characteristics of the noise.

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